there is no mismatch. Although the test equipment was not capable of resolving an effect smaller than 2%, the nature of the results strongly indicated that the effect was even smaller.

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COMMENTS

Comment on "Satellite Motions About an Oblate Planet"

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THE solution presented in Ref. 1 is valid only for a time interval of order unity since θ_1 contains the mixed secular term

$$\frac{-R^2 \sin 2i}{2c^4 r_0^2}\, \xi \cos\xi$$

In this note it will be shown that this secular term can be eliminated by expressing the solution in a frame that rotates at an appropriate uniform rate about the polar axis. This technique was used[2] for satellite motions in the restricted three-body problem.

Consider the coordinate system x^*,y^*,z^* defined with respect to the X,Y,Z system of Ref. 1 by

$$\begin{aligned} x^* &= X \cos\Omega + Y \sin\Omega \\ y^* &= -X \cos i \sin\Omega + Y \cos i \cos\Omega + Z \sin i \\ z^* &= X \sin i \sin\Omega - Y \sin i \cos\Omega + Z \cos i \end{aligned} \qquad (1)$$

In Eqs. (1), Ω is the longitude of the node that is assumed to rotate slowly in the X-Y plane [cf., discussion below Eq. (10)].

If the analysis of Ref. 1 is paralleled using the starred variables of Eqs. (1) instead of the X,Y,Z variables, one can derive the three equations corresponding to (3.15) of Ref. 1. In the present context r,θ,φ,u, and P are all formally related to starred variables by the same equations used in Ref. 1. Since Ω now depends on time, the following additional terms will appear on the left-hand sides of Eqs. (3.15):

Equation for u

$$(2 \cos i \sin^2\theta + 2 \cos\varphi \sin i (d\theta/d\varphi) - \sin 2\theta \sin\varphi \sin i) Pu(d\Omega/d\varphi) + O(J^2) \qquad (2)$$

Equation for θ

$$(d/d\varphi)[P(d\Omega/d\varphi)\cos\varphi \sin i] - (\cos i \sin 2\theta - \cos 2\theta \sin\varphi \sin i)P(d\Omega/d\varphi) + O(J^2) \qquad (3)$$

Equation for P

$$(d/d\varphi)\{[P(d\Omega/d\varphi)](\cos i \sin^2\theta - \tfrac{1}{2}\sin 2\theta \sin\varphi \sin i)\} + (\sin\varphi \sin i(d\theta/d\varphi) + \tfrac{1}{2}\sin 2\theta \cos\varphi \sin i)P(d\Omega/d\varphi) + O(J^2) \qquad (4)$$

In Ref. 1 no distinction was made between the variable $\varphi = (1 + J\varphi_1)\xi$ and ξ in the arguments of the trigonometric terms. It is clear that when $\xi = O(1/J)$ (i.e., for large times) such an approximation cannot be valid and will not be made here. In contrast, it is pointed out that trigonometric terms with the argument θ may be expanded since θ_1 will later be shown to be a bounded function of ξ.

Assuming the asymptotic expansions for the three variables u, θ, and P of Ref. 1, one obtains

$$u_0'' + u_0 = \mu/P_0^2 \qquad (5)$$

$$\theta_0 = \pi/2 \qquad (6)$$

$$P_0' = 0 \qquad (7)$$

$$u_1'' + u_1 = -2\varphi_1 u_0 - 2u_0\omega \cos i + (\mu/P_0^2)\{2\varphi_1 - (2P_1/P_0) + R^2[u_0^2 + (u_0 u_0' \sin 2\varphi)\sin^2 i - (3u_0^2 \sin^2\varphi)\sin^2 i]\} \qquad (8)$$

$$\theta_1'' + \theta_1 = (\mu R^2 u_0/P_0^2) \sin 2i \sin\varphi + 2\omega \sin\varphi \sin i \qquad (9)$$

$$P_1' = -(\mu R^2 u_0/P_0)\sin 2\varphi \sin^2 i \qquad (10)$$

where primes denote differentiation with respect to ξ, and $d\Omega/d\varphi = J\omega$ for some constant ω to be determined later.

If at time $t = 0$, the following general initial conditions apply

$$r = r_0 \quad \theta = \frac{\pi}{2} \quad \varphi = \varphi_0 \qquad \frac{dr}{dt} = 0 \quad \frac{d\theta}{dt} = 0 \quad \frac{d\varphi}{dt} = \frac{V_0}{r_0} \qquad (11)$$

(Note that V_0, r_0, φ_0 are now measured in the moving frame.) The corresponding initial values when $\varphi = \varphi_0$ or $\xi = \xi_0 \equiv \varphi_0(1 - J\varphi_1)$ are

$$\begin{array}{lll} u_0 = 1/r_0 & \theta_0 = \pi/2 & P_0 = r_0 V_0 \\ u_0' = 0 & \theta_0' = 0 & \\ u_1 = 0 & \theta_1 = 0 & P_1 = 0 \\ u_1' = 0 & \theta_1' = 0 & \end{array} \qquad (12)$$

The solution $\theta_0 = \pi/2$ given in (6) follows from the limiting equation for θ as $J \to 0$, and the initial conditions $\theta_0 = \pi/2$ and $\theta_0' = 0$.

The solutions of Eqs. (5, 7, and 10) are

$$u_0 = K[1 + \eta \cos\psi] \qquad (13)$$

$$P_0 = r_0 V_0 \qquad (14)$$

$$P_1 = (-R^2K^2P_0 \sin^2 i/6)[(3 + 4\eta)\cos 2\varphi_0 - 3\eta \cos(2\varphi - \psi) - 3\cos 2\varphi - \eta\cos(2\varphi + \psi)] \qquad (15)$$

where

$$K = \mu/P_0^2 = 1/r_0 c^2$$

$$\psi = \xi - \xi_0$$

When the solution for u_0 given by (13) is substituted into (9) one obtains

$$\theta_1'' + \theta_1 = (K^2R^2\, \eta \sin 2i/2)[\sin(\varphi + \psi) + \sin(\varphi - \psi)] + \delta \sin\varphi \qquad (16)$$

where

$$\delta = 2\,[K^2R^2 \cos i + \omega] \sin i$$

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Note that Eq. (16) contains no true secular term with the argument ξ; this is a coincidence that needs not prevail to higher orders.

The solution of (16), subject to the initial conditions for θ_1, is

$$\theta_1 = \frac{K^2R^2\eta \sin 2i}{6} [2\sin(\psi - \varphi_0) - \sin(\varphi + \psi) + 3\sin(\varphi - \psi)] + \frac{\delta}{2}\cos\varphi_0 \sin\psi - \frac{\delta}{J\varphi_1}\sin\left(\frac{J\varphi_1\psi}{2}\right) \times \cos\left(\varphi - \frac{J\varphi_1\psi}{2}\right) \quad (17)$$

The last term of Eq. (17) comprises the response to the almost resonant forcing function $\delta \sin\varphi$ and part of the complementary solution. If $\varphi_1 \neq 0$, this term describes a beating oscillation whose amplitude is $O(1/J)$ in violation of the assumed form for the asymptotic expansion of θ. Furthermore, in the limit as $\varphi_1 \to 0$ the term in question tends to $-(\delta/2)\psi \cos\xi$ which is unbounded. This difficulty is easily averted by choosing $\delta = 0$, i.e., $\omega = -K^2R^2 \cos i$. This means that if the node rotates at the just-mentioned rate the representation for θ is bounded to order J in this rotating frame.

When the known functions appearing on the right-hand side of Eq. (8) are substituted, it can be shown that all terms with the argument $\varphi = (1 + J\varphi_1)\xi$ drop out, another occurrence that needs not hold to higher orders. The true secular term with the argument ξ can be eliminated by choosing

$$\varphi_1 = 2K^2R^2 (1 - \tfrac{5}{4}\sin^2 i)$$

and u_1 can be expressed in terms of bounded functions in the form

$$u_1 = (K^3R^2/24)\{[-(8 + 15\eta + 16\eta^2)\sin^2 i \sin 2\varphi_0] \sin\psi + [-(72 + 8\eta^2) + (84 + 12\eta^2)\sin^2 i + (-20 - 27\eta + 4\eta^2)\sin^2 i \cos 2\varphi_0]\cos\psi - (4 + 6\eta^2)\sin^2 i \cos 2\varphi + 2\eta^2(3\sin^2 i - 2)\cos 2\psi - 5\eta \sin^2 i \cos(2\varphi + \psi) - \eta^2 \sin^2 i \cos 2(\varphi + \psi) + 3\eta^2 \sin^2 i \cos 2(\varphi - \psi) + 72 + 12\eta^2 - (84 + 18\eta^2)\sin^2 i + (24 + 32\eta)\sin^2 i \cos 2\varphi_0\} \quad (18)$$

The just mentioned asymptotic expansions for u and θ are bounded to order J only because Eq. (16) does not contain a term with the argument ξ, and in the differential equation for u_1 the terms with the argument φ canceled out.

In Ref. 3 it is pointed out that the solution to order J^2 for this problem breaks down at the "critical inclination" $i = \sin^{-1} 2/5^{1/2}$ (which corresponds to $\varphi_1 = 0$), and this question has been discussed in the subsequent literature.

The extension of the present method to order J^2 undoubtedly will lead to the same difficulty, since there is no guarantee that terms with the argument ξ as well as terms with the argument φ can be eliminated both in the u_2 and θ_2 equations through the choice of only two additional arbitrary constants ω_2 and φ_2.

It is not surprising that the method of Lindstedt will fail to order J^2 since this method is strictly applicable to the case of *periodic* solutions. In order to derive an asymptotic solution valid for large times for an arbitrary initial value problem of this type it is necessary to use the more general techniques discussed in the literature (cf., Ref. 4).

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Comments on "Prediction and Measurement of Natural Vibrations of Multistage Launch Vehicles"

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In their interesting paper,[1] Alley and Leadbetter present an analytic procedure for finding the natural modes of free-free multistage launch vehicles. A somewhat simpler procedure for finding these free-free modes has been described[2] previously and consists in iterating the matrix equation

$$[R][C][m]\{w_r\} = (1/\omega^2)\{w_r\}$$

where $[C]$ and $[m]$ are the influence coefficient and diagonal mass matrices respectively, and $[R]$ for these free-free missiles is given by

$$[R] = [[1] - (1/M)\{1\}\lfloor m \rfloor - (1/I_y)\{\bar{x}\}\lfloor m\bar{x} \rfloor]$$

All square matrices are of order p, where p is the number of masses. In the forementioned, $\bar{x}$ is distance measured from the center of gravity. It can be shown readily that the elements of $[R]$ just mentioned are identical to those of the $[B]$ matrix of Ref. 1. Furthermore, it was shown in Ref. 2 that the $[C]$ matrix can be any set of influence coefficients for the structure restrained in any way—cantilever, simply-supported, and at any arbitrary points, etc. The restrictions on the influence coefficients mentioned in Ref. 1 are not necessary here, and any convenient set can be used.

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Comment on "The Shock Stand-Off Distance with Stagnation-Point Mass Transfer"

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The Appendix of Ref. 1 contains an analysis of the title subject based on an extension of Lighthill's[2] solution for inviscid hypersonic flow around spheres. The purpose of

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